Dynamical Analysis of a Predator-Prey System under Harvesting, Competition and Gestational Delay

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Abstract

In the present paper a predator-prey model have been considered where Holling type II functional response is used. Harvesting of both the species have been considered. The growth function followed by the prey population is the logistics law of growth function. Competition among the predator population is also considered. Gestational delay of the predator population is discrete-type. The existence of steady states and stability of the proposed model has been discussed. The stability analysis was carried out using the eigenvalue analysis. Harvesting the ecological population is an important aspect, and the existence of bionomic equilibrium of the model has been studied. To determine the optimal harvesting policy Pontryagin's maximum principle is used. The gestational delay is incorporated into the predator species, and the delay's effect on the model's dynamic behaviour is analysed. The relevant properties of the delay model are taken care of. This study reflects the appearance of Hopf-bifurcation, leading to the periodic oscillation of the populations.

Keyword: prey-predator, nonlinear differential equation, stability, bionomic equilibrium, gestational delay, Hopf bifurcation.

1 Introduction and Motivation

In the ecological system, the population size depends on many factors. Prey-predator interaction among species plays an important role. Mathematical models in the form of nonlinear differential equations are used for the discussion of dynamical behaviour of the interacting populations. The dynamic behaviour of problems related to harvesting of species is a very interesting research topic. These problems have attracted the researchers for many years. Clark (1976,1985) [1, 2] has discussed in his book the prospect of bio-economic modelling involving multi-species fisheries. Lotka-Volterra [3, 4] first modelled the prey-predator model, where they considered a response function which vary proportionaly to the number of predators.

A simple model of prey-predator interaction that Lotka and Volterra proposed is:

$$\begin{aligned} \frac{dx}{dt} &= x(r-\alpha y), \\ \frac{dy}{dt} &= y(-s+\beta x). \end{aligned} \tag{1.1}$$

Here r, s, α, β all are positive and are called logistic parameters. Various aspects of this model have been discussed in the ecological literature. Here the law of logistic growth is follwed by the prey population. Normally, different growth functions are followed in the prey-predator model. To exist in an ecosystem for the long term, response functions between prey-predators have an important role. Holling-type response functions [5] are some of them. In ecology, the stability of the ecological system is our fundamental concern. Different mathematical models are used for the investigation of the stability of ecological systems.

Holling proposed functional responses which are Holling type I, type II and type III functional response. There are several literal using the Holling type functional responses. Kar, Chakraborty and Pahari [6] considered a prey-predator model where Holling type II functional response is utilised. In this model harvesting of each species is considered. They have discussed the effect of harvesting to control the system. Das, Mukherjee and Chaudhuri [7] studied a prey-predator problem with non-selective harvesting.

There are several literature on a predator-prey model describing the impact of time delay. Toaha et al [8], Gazi et al. [9] studied a system consisting of time delay. They have found the necessary conditions of harvesting for stable equilibrium points. They have shown that instability and presence of Hopf bifurcation can be induced by the time delay . They have shown that there exists some critical value of the effort for which the profit is maximum and equilibrium point remain stable. Dai and Tang [10], Xiao and Ruan [11] examined a predator-prey model where rate of harvesting is constant. Feng [12] studied a delayed model involving three species. He studied the effect of growth rate and interaction rates between the species . A predator prey model involving delay is studied by Nakaoka et al [13]. They have shown that, for small delay, the globally asymptotically stability of the system is maintained where as chaotic behaviour is exhibited for large delay. Yafia [14] established an explicit algorithm to determine the direction of Hopf bifurcation of a model with single delay. Kar and Matsuda [15], and Kar and Pahari [16] considered a predator-prey model in presence of time delay and examined it's effect on the dynamics of the model. They have shown that the time delay has effect in changing the position of stable equilibrium points.

Under the above review of the works in the field of population biology, we take an ecological system. We organize our work as follows: In section 2, formulation of model system is described. Two differential equations for prey and predator populations are taken. In section 3, we discuss the preliminary results of the system namely, the existence of solution, boundedness, existence of equilibrium and the threshold conditions. Section 4 studies the stability of the system without delay. Section 5 describes analysis for harvesting, namely bionomic equilibrium and optimal harvesting policy. Section 6 describes the analysis of the delayed system including stability and periodic oscillation due to Hopf bifurcation. The last section discusses the entire analysis, including numerical simulation and the study's conclusion.

2 The Model Formulation

We have considered a dynamical model with one predator and one prey where the prey population follow law of logistic growth as follows:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} \\ \frac{dy}{dt} &= \frac{\gamma_1 xy}{\mu + x} - dy \end{aligned} \tag{2.1}$$

where $d, \alpha, \beta, \gamma, \gamma_1$ are positive constants and $\gamma \geq \gamma_1$. $x(t)$ represents prey population and $y(t)$ predator population densities at any time t.

For earning capital, harvesting is used. Uncontrolled harvesting may affect the ecological system. Incorporating hervesting of both the prey and predator populations with harvesting efforts E_1 and E_2 respectively we may write the above problem as

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} - q_1 E_1 x \\ \frac{dy}{dt} &= \frac{\gamma_1 xy}{\mu + x} - dy - q_2 E_2 y \end{aligned} \tag{2.2}$$

And one more common phenomenon is the competition among the predator population for common resources. We have incorporated the crowding effect. For this, we shall add an extra removal term $-\beta_1 y^2$ with β_1 as the competition coefficient in the dynamics of predator population. So, finally, the predator-prey dynamical system is:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} - q_1 E_1 x \\ \frac{dy}{dt} &= \frac{\gamma_1 xy}{\mu + x} - dy - \beta_1 y^2 - q_2 E_2 y \end{aligned} \tag{2.3}$$

where $d, \alpha, \beta, \beta_1, \gamma, \gamma_1, q_1, q_2, E_1, E_2$ are positive constants q_1 and q_2 are catchability coefficients, and d is the death rate of the predator.

For simplicity of the calculations, we take $q_1 = q_2 = 1$ and the system (2.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} - E_1 x \\ \frac{dy}{dt} &= \frac{\gamma_1 xy}{\mu + x} - dy - \beta_1 y^2 - E_2 y \end{aligned} \tag{2.4}$$

We shall consider this predator-prey system under harvesting and a time delay that occur due to gestation. The energy of the predator obtained from the prey population as food will enhance the population of the predator . The consumption of the prey population and reproduction of the predator are not instantaneous. So there is some time interval between the prey hunting and the addition of biomass to the predator population. Let us consider τ as a time delay. Considering this time delay the model becomes

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} - E_1 x \\ \frac{dy}{dt} &= \frac{\gamma_1 x(t-\tau)y}{\mu + x(t-\tau)} - dy - \beta_1 y^2 - E_2 y \end{aligned} \tag{2.5}$$

with the initial condition $x(\theta) \geq 0$, $y(\theta) \geq 0$, $\theta\epsilon[-\tau, 0)$, and $x(0), y(0)$ are positive. Our aim is to study two types of stability (a) stability which is independent of delay and (b) stability depending on the delay.

3 Some basic results

Let us now study the absolute stability of the above dynamical system. For τ=0 the system changes to:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} - E_1 x \\ \frac{dy}{dt} &= \frac{\gamma_1 xy}{\mu + x} - dy - \beta_1 y^2 - E_2 y \end{aligned} \tag{3.1}$$

The basic results of the model system will be discussed here. Many dynamic conditions are incorporated into the model system. The proposed model is a predator-prey model. Self-crowding of the predator population is incorporated. Both the species are harvested. The rate of harvesting is variable.

3.1 Existence and positive invariance

Here, $f_1 = \alpha x - \beta x^2 - \frac{\gamma xy}{\mu+x} - E_1 x$, $f_2 = \frac{\gamma_1 xy}{\mu+x} - dy - \beta_1 y^2 - E_2 y$ are the smooth functions of variables x, y in the positive quadrant $\{(x, y) : x \geq 0, y \geq 0\}$. Therefore, the solutions of the system exist and they are unique

3.2 Boundedness

Theorem: All the solutions of the proposed model in R_+^2 are uniformly bounded.

Proof: Let us consider any solution of the system with initial value $x(0) > 0, y(0) > 0$ be $(x(t), y(t))$. Let $W = x + \frac{\gamma}{\gamma_1}y$. Then, $\frac{dW}{dt} + dW = (\alpha - E_1 + d)x - \beta x^2 - \frac{\gamma}{\gamma_1}E_2 y - \frac{\gamma}{\gamma_1}\beta_1 y^2$ $\leq (\alpha - E_1 + d)x - \beta x^2 \leq \frac{(\alpha - E_1 + d)^2}{4\beta} = L$.

Now using theory of inequality [17], we get,

$$0 \leq W(x, y) \leq \frac{L}{\delta} + \frac{W(x(0), y(0))}{e^{\delta t}} \tag{3.2}$$

For t tends to infinity, we get $0 \leq W(x, y) \leq \frac{L}{\delta}$. Thus all solutions of the system which start in R_2^+ will enter in the region $R = \{(x, y) : 0 \leq W \leq \frac{L}{\delta} + \epsilon, \forall \epsilon > 0\}$. Hence the theorem is proved.

3.3 Equilibrium Points

To investigate the dynamical behaviour of the prey-predator system now we shall analyze the model system which we have considered. Now we want to get some information about the population in future. Here, we solve the following equations to get the fixed points or steady states of the system (3.1).

$$\begin{aligned} \alpha x - \beta x^2 - \frac{\gamma x y}{\mu + x} - E_1 x &= 0 \\ \frac{\gamma_1 x y}{\mu + x} - dy - \beta_1 y^2 - E_2 y &= 0 \end{aligned} \tag{3.3}$$

(a)$P_0(0, 0)$ satisfies the null-clines, so (0,0) is the trivial equilibrium point.

(b) When y=0, we get $x = \frac{\alpha - E_1}{\beta}$. x is positive when $\alpha > E_1$.

Thus $P_1(x_1, 0)$ is an axial equilibrium point where $x_1 = \frac{\alpha - E_1}{\beta}$.

This axial equilibrium point exists only when growth rate of prey population is greater than the harvesting rate of the population and the predator population will extinct from the system.

(c) The interior equilibrium point is $P_2(x_2, y_2)$ satisfies the following equations

$$\alpha - \beta x - \frac{\gamma y}{\mu + x} - E_1 = 0$$

$$\frac{\gamma_1 x}{\mu + x} - d - \beta_1 y - E_2 \quad = \quad 0 \tag{3.4}$$

Here $y_2 = \frac{1}{\beta_1}[\frac{\gamma_1 x_2}{\mu + x_2} - (d - E_2)]$ and x_2 is given by a 3rd-degree polynomial equation:

$$Ax_2^3 + Bx_2^2 + Cx_2 + D = 0, \tag{3.5}$$

where $A = \beta\beta_1 > 0$, $B = \beta_1(2\mu\beta - (\alpha - E_1))$, $C = \beta\beta_1\mu^2 - 2\mu\beta_1(\alpha - E_1) + \gamma\gamma_1 - \gamma(d + E_2)$, $D = -\beta_1\mu^2(\alpha + E_1) - \gamma(d + E_2) < 0$.

By Descartes' rule of sign the above equation will have at least one positive root. Thus at least one positive value of x_2 exists. For this value of x_2 , $y_2 = \frac{1}{\beta_1}[\frac{\gamma_1 x_2}{\mu + x_2} - (d - E_2)]$, which is positive if $x_2 > \frac{\mu(d+E_2)}{\gamma_1 - (d+E_2)}$ and $\gamma_1 > d + E_2$. Thus the interior equilibrium point exists and it depends on the parameters involved in the system equations.

4 Stability Analysis without delay

We are at the stage where the local behaviour near the fixed points will be analysed . The stability of the nonlinear model system near these points depends on the eigenvalues of the Jacobian matrix of the corresponding linear system [18]. After linearizing the above nonlinear system we shall determine the Jacobian matrix. At any point (x,y) the Jacobian matrix of the system of (2.4) is:

$$J = \begin{pmatrix} \alpha - 2\beta x - \gamma y \frac{\mu}{(\mu + x)^2} - E_1 & -\frac{\gamma x}{\mu + x} \\ \frac{\gamma_1 y \mu}{(\mu + x)^2} & \frac{\gamma_1 x}{\mu + x} - 2\beta_1 y - d - E_2 \end{pmatrix}. \tag{4.1}$$

At $P_0(0, 0)$, the Jacobian matrix is:

$$\begin{pmatrix} \alpha - E_1 & 0 \\ 0 & -(d + E_2) \end{pmatrix}.$$

From the above matrix, it is clear that the eigenvalues are negative if $\alpha < E_1$. This shows that if the intrinsic growth rate of the prey species is less than the harvesting effort then the trivial equilibrium point $P(0, 0)$ is stable biologically which implies that the prey population will extinct from the system.

At the predator free equilibrium point $P_1(x_1, 0)$, the Jacobian matrix is:

$$\begin{pmatrix} -(\alpha - E_1) & -\frac{\gamma(\alpha - E_1)}{\beta\mu + \alpha - E_1} \\ 0 & \frac{\gamma_1(\alpha - E_1)}{\beta\mu + \alpha - E_1} - d - E_2 \end{pmatrix}.$$

The eigenvalues are $E_1 - \alpha$ and $\frac{\gamma_1(\alpha - E_1)}{\beta\mu + \alpha - E_1} - d - E_2$. The equilibrium point $P_1(x_1, 0)$ is stable equilibrium point if $E_1 < \alpha$ and $\frac{\gamma(\alpha - E_1}{\beta\mu + \alpha - E_1} - d < E_2$. That is, stability occurs when the harvesting effort is less than the growth rate of prey population, also the combined effort of death rate and harvesting effort of the predator population is greater than a positive value, which depends on the prey population present after harvesting.

At $P_2(x_2, y_2)$, the Jacobian matrix J_2 is:

$$\begin{pmatrix} -\beta x_2 + \frac{\gamma x_2 y_2}{(\mu + x_2)^2} & -\frac{\gamma x_2}{\mu + x_2} \\ \frac{\gamma_1 \mu y_2}{(\mu + x_2)^2} & -\beta_1 y_2 \end{pmatrix}.$$

The characteristic equation of the above matrix is

$$\lambda^2 - (TrJ_2)\lambda + DetJ_2 = 0, \tag{4.2}$$

where

$TrJ_2 = -\beta x_2 - \beta_1 y_2 + \gamma \frac{x_2 y_2}{(\mu + x_2)^2}$ and $Det(J_2) = x_2 y_2[\beta\beta_1 - \frac{\gamma\beta_1 y_2}{(\mu + x_2)^2} + \frac{\gamma\gamma_1\mu}{(\mu + x_2)^3}]$. For stable equilibrium point $Tr(J_2) < 0$ and $Det(J_2) > 0$. $Tr(J_2) < 0$ implies $\beta_1 > \frac{\gamma x_2}{(\mu + x_2)^2} - \frac{\beta x_2}{y_2}$ and $Det(J_2) > 0$ implies $\beta_1 < \frac{\gamma\gamma_1\mu}{(\mu + x_2)(\gamma y_2 - \beta(\mu + x_2)^2}$, also $\frac{\beta}{\gamma} < \frac{y_2}{(\mu + x_2)^2}$.

Theorem: The coexistence equilibrium point is stable if $\frac{\gamma x_2}{(\mu + x_2)^2} - \frac{\beta x_2}{y_2} < \beta_1 < \frac{\gamma\gamma_1\mu}{(\mu + x_2)(\gamma y_2 - \beta(\mu + x_2)^2)}$ and $\frac{\beta}{\gamma} < \frac{y_2}{(\mu + x_2)^2}$.
Thus, the stability of the system depends on crowding of predator species, death rate, predation rate, conversion rate and harvesting effort of the predator species.

4.1 Nature of periodic solution

Here β_1 is the parameter representing the crowding effect of the predator population. Due to the change of this parameter, the stable solution bifurcates into a periodic oscillation. The characteristic equation gives two purely imaginary eigenvalues if the $TrJ_2 = 0$ and $DetJ_2 > 0$. When $\beta_1 = \frac{\gamma x_2}{(\mu + x_2)^2} - \frac{\beta x_2}{y_2} = \beta_1^*$ the characteristic equation gives two purely imaginary root provided that $\frac{\beta}{\gamma} < \frac{y_2}{(\mu + x_2)^2}$. And also $\frac{d\Re\lambda}{d\beta_1}|_{\beta_1 = \beta_1 *} \neq 0$. These are the if and only if conditions for the existence of Hopf bifurcation (Liu 1994)[19]. When $\beta_1 = \beta_1 *$ the characteristic equation reduces to $\lambda^2 = DetJ$. Since $DetJ$ is positive,

$Re\lambda|_{\beta_1=\beta_1*} = \sqrt{DetJ} \neq 0$. Thus, both conditions are satisfied. Therefore it is clear that system undergoes a Hopf bifurcation for $\beta_1 = \beta_1*$.

Now nature of Hopf bifurcating periodic solutions will be determined. To determine the nature we calculate the Liapunov number following Perko [20] at β_1^*.

Let us introduce small perturbations $x = \xi + x^*|_{\beta_1=\beta_1^*}$, $y = \eta + y^*|_{\beta_1=\beta_1^*}$ in (3.1) and then using Taylor series expansion, we have

$$\begin{aligned} \dot{\xi} &= a_{10}\xi + a_{01}\eta + a_{20}\xi^2 + a_{11}\xi\eta + a_{30}\xi^3 + a_{21}\xi^2\eta + ..., \\ \dot{\eta} &= b_{10}\xi + b_{01}\eta + b_{20}\xi^2 + b_{11}\xi\eta + b_{30}\xi^3 + b_{21}\xi^2\eta + ... \end{aligned} \quad (4.3)$$

where $a_{10}, a_{01}, b_{10}, b_{01}$ are the components of the Jacobian matrix at the equilibrium point β_1^*. Hence $a_{10} + b_{01} = 0$ and $\Delta = a_{10}b_{01} - a_{01}b_{10} = DetJ > 0$. The expression of the coefficients a_{ij} and b_{ij} at (x_2, y_2, β_1^*) are given below:

$a_{10} = \frac{\partial f_1}{\partial x} = \alpha - 2\beta x_2 - \frac{\mu\gamma y_2}{(\mu+x_2)^2} - q_1E_1; a_{01} = \frac{\partial f_1}{\partial y} = -\frac{\gamma x_2}{\mu+x_2}; a_{11} = \frac{1}{2}\frac{\partial^2 f_1}{\partial x\partial y} = -\frac{\gamma\mu}{2(\mu+x_2)^2}$

$a_{20} = \frac{1}{2}\frac{\partial^2 f_1}{\partial x^2} = \frac{\mu\gamma y_2}{(\mu+x_2)^3} - \beta; a_{02} = \frac{1}{2}\frac{\partial^2 f_1}{\partial y^2} = 0; a_{12} = \frac{1}{2}\frac{\partial^3 f_1}{\partial x\partial y^2} = 0; a_{21} = \frac{1}{2}\frac{\partial^2 f_1}{\partial x^2\partial y} = \frac{\mu\gamma}{(\mu+x_2)^3}$

$a_{30} = \frac{1}{6}\frac{\partial^3 f_1}{\partial x^3} = -\frac{\mu\gamma y_2}{(\mu+x_2)^4}; b_{01} = \frac{\partial f_2}{\partial y} = \frac{\gamma_1 x_2}{\mu+x_2} - d - 2\beta_1 y_2 - q_2E_2; b_{10} = \frac{\partial f_2}{\partial y} = \frac{\mu\gamma_1 y_2}{(\mu+x_2)^2}$

$b_{11} = \frac{1}{2}\frac{\partial^2 f_2}{\partial x\partial y} = \frac{\gamma_1\mu}{2(\mu+x_2)^2}; b_{20} = \frac{1}{2}\frac{\partial^2 f_2}{\partial x^2} = -\frac{\mu\gamma_1 y_2}{(\mu+x_2)^3}; b_{02} = \frac{1}{2}\frac{\partial^2 f_2}{\partial y^2} = -\beta_1$

$b_{12} = \frac{1}{2}\frac{\partial^3 f_2}{\partial x\partial y^2} = 0; b_{21} = \frac{1}{2}\frac{\partial^2 f_2}{\partial x^2\partial y} = -\frac{\mu\gamma_1}{(\mu+x_2)^3}; b_{30} = \frac{1}{6}\frac{\partial^3 f_2}{\partial x^3} = \frac{\mu\gamma_1 y_2}{(\mu+x_2)^4}$.

Thus, the expression of first Liapunov number (cf. Perko [20]) is given by

$$\begin{aligned} \sigma &= -\frac{3\pi}{2a_{10}\Delta}[a_{10}b_{01}a_{11}^2 + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11}) - 2a_{10}a_{01}a_{20}^2 - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) - a_{11}a_{20} \\ &\quad \times(a_{01}b_{10} - 2a_{10}^2) - (a_{10}^2 + a_{01}b_{10}) - 3a_{01}a_{30} + 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{01}b_{21})] \end{aligned} \quad (4.4)$$

Using the above a_{ij} in (6.2), we have

$$\begin{aligned} \sigma &= -\frac{3\pi}{2a_{10}\Delta}[a_{10}b_{01}a_{11}^2 + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11}) - 2a_{10}a_{01}a_{20}^2 - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) - a_{11}a_{20} \\ &\quad \times(a_{01}b_{10} - 2a_{10}^2) - (a_{10}^2 + a_{01}b_{10}) - 3a_{01}a_{30} + 2a_{10}a_{21} - a_{01}b_{21}]. \end{aligned} \quad (4.5)$$

For $\sigma < 0$, the equilibrium point P_2 is supercritical Hopf-bifurcation. For $\sigma > 0$, the equilibrium point P_2 is subcritical Hopf bifurcation. If the bifurcation is supercritical, then the periodic orbits are stable, otherwise, they are unstable.

5 Bionomic analysis

Here we shall discuss about bionomic equilibrium of proposed model. Harvesting is an important aspect of the system but there must be limitation. Unlimited or uncontrolled harvesting may affect the ecosystem. The balance between biological equilibrium and economic equilibrium is bionomic equilibrium. From economic point of view we want to make the total revenue from harvested biomass greater or equal to the total cost of effort. If c_1, c_2 are fishing cost of prey and predator respectively for unit effort and p_1, p_2 are price of prey and predator respectively for unit biomass

Then net income at any time is $(p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2$. Then the bionomic equilibrium can be determined from the following set of equations:

$$\begin{aligned} \alpha x - \beta x^2 - \frac{\gamma xy}{\mu + x} - q_1E_1x &= 0, \\ \frac{\gamma_1 xy}{\mu + x} - dy - \beta_1 y^2 - q_2E_2y &= 0, \\ (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 &= 0. \end{aligned} \tag{5.1}$$

The bionomic equilibrium fishing cost must be less than the revenue for the predator or for the prey or for both the predator and prey. We shall take $E_2 = 0$ when the predator population fishing cost is more than the revenue. We take $E_1 = 0$ when the prey population fishing cost is more than the revenue. This means that the fishing is not feasible.

When $E_2 = 0$ and $E_1 \neq 0$, we have $x_\infty = \frac{c_1}{p_1q_1}$, $y_\infty = \frac{\mu+\frac{c_1}{p_1q_1}}{\gamma}[\alpha - \frac{\beta c_1}{p_1q_1} - q_1E_1]$.

When $E_1 = 0$ and $E_2 \neq 0$, we have $y_\infty = \frac{c_2}{p_2q_2}$, $x_\infty = \frac{\mu(d+\frac{\beta_1c_2}{p_2q_2}+q_2E_2)}{\gamma_1-(d+\frac{\beta_1c_2}{p_2q_2}+q_2E_2)}$.

When $E_1 \neq 0$ and $E_2 \neq 0$, we have $x_\infty = \frac{c_1}{p_1q_1}$, $y_\infty = \frac{c_2}{p_2q_2}$, $E_{1\infty} = \frac{1}{q_1}[\alpha - \frac{\beta c_1}{p_1q_1} - \frac{\gamma c_2}{p_2q_2(\mu+\frac{c_1}{p_1q_1})}]$, $E_{2\infty} = \frac{1}{q_2}[-d + \frac{\gamma_1c_1}{\mu p_1q_1+c_1} - \frac{\beta_1c_2}{p_2q_2}]$.

For existence of the non-trivial bionomic equilibrium point, we must have $E_{1\infty} > 0$, $E_{2\infty} > 0$, which hold only when

$$\alpha > \frac{\beta c_1}{p_1q_1} + \frac{\gamma c_2}{p_2q_2(\mu + \frac{c_1}{p_1q_1})}, \qquad d < \frac{\gamma_1c_1}{\mu p_1q_1 + c_1} - \frac{\beta_1c_2}{p_2q_2}.$$

5.1 Optimal harvesting policy

In the context of harvesting, our aim is to maximize some functions related to the problem. In cases where the population are harvested for profit, it is seen that over harvesting leads to the decline of the entire population. The population may extinct from the system. This action results in destroying the balance of the stable ecosystem. Here, we want to maximize the present value J over some time interval $[0,\infty]$ subject to the constraints (5.1). The problem now is a control system problem with objective functional

$$J = \int_0^\infty [(p_1 q_1 x - c_1)E_1 + (p_2 q_2 y - c_2)E_2] e^{-\delta_1 t} dt \tag{5.2}$$

subject to the constraints as stated above, where δ_1 is the instantaneous annual discount rate. Our aim is to maximize the objective functional (Net revenue) using Pontryagin's maximum principle. Here $E_1(t)$ and $E_2(t)$ are control variables and they will satisfy the constraints $0 \leq E_1(t) \leq (E_1)_{max}$ and $0 \leq E_1(t) \leq (E_1)_{max}$. We now construct the Hamiltonian H of the problem as follows:

$$H = e^{-\delta_1 t}[(p_1 q_1 x - c_1)E_1 + (p_2 q_2 y - c_2)E_2] + \lambda_1(t)[\alpha x - \beta x^2 - \frac{\gamma x y}{\mu + x} - q_1 E_1 x] + \lambda_2(t)[\frac{\gamma_1 x y}{\mu + x} - dy - \beta_1 y^2 - q_2 E_2 y] \tag{5.3}$$

Adjoint variables are λ_1 and λ_2. Upper limit of the harvesting effort is E_{max} . Let optimal control be E when x, y are the corresponding responses of the system. Then Pontryagin's maximum principle gives,

$$\frac{\partial H}{\partial E_1} = 0, \frac{\partial H}{\partial E_2} = 0, \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}. \tag{5.4}$$

$\frac{\partial H}{\partial E_1} = 0$ gives $\lambda_1 = e^{-\delta t}(p_1 - \frac{c_1}{q_1 x_2})$,
$\frac{\partial H}{\partial E_2} = 0$ gives $\lambda_2 = e^{-\delta t}(p_2 - \frac{c_2}{q_2 y_2})$.
Also we have

$\frac{d\lambda_1}{dt} + A\lambda_1 = -e^{-\delta t}B$, $\frac{d\lambda_2}{dt} + C\lambda_2 = -e^{-\delta t}D$.

Where A, B, C, D are given by:

$A = \alpha - 2\beta x_2 - \frac{\gamma \mu x_2}{(\mu + x_2)^2} - q_1 E_1, B = p_1 q_1 E_1 + \frac{\gamma_1 \mu y_2}{(\mu + x_2)^2}(p_2 - \frac{c_2}{q_2 y_2})$,
$C = \frac{\gamma_1 x_2}{\mu + x_2} - d - 2\beta_1 y_2 - q_2 E_2, D = p_2 q_2 E - \frac{\gamma x_2}{\mu + x_2}(p_1 - \frac{c_1}{q_1 x_2})$.

Thus, the solution for the adjoint variables λ_1 and λ_2 are given by
$\lambda_1 = -\frac{B}{A-\delta}e^{-\delta t}$ and $\lambda_2 = -\frac{D}{C-\delta}e^{-\delta t}$.

Comparing the values of λ_1 and λ_2, the singular paths are
$p_1 - \frac{c_1}{q_1 x_2} + \frac{B}{A-\delta} = 0$, $p_2 - \frac{c_2}{q_2 y_2} + \frac{D}{C-\delta} = 0$.

For positive root $x_2 = x_{2\delta}, y_2 = y_{2\delta}$, we get
$E_{1\delta} = \frac{1}{q_1}[\alpha - \frac{\beta c_1}{p_1 q_1} - \frac{\gamma c_2}{p_2 q_2(\mu + \frac{c_1}{p_1 q_1})}]$, $E_{2\delta} = \frac{1}{q_2}[-d + \frac{\gamma_1 c_1}{\mu p_1 q_1 + c_1} - \frac{\beta_1 c_2}{p_2 q_2}]$.

Thus, the optimal harvesting effort can be determined when the optimal equilibrium is determined. The values of $\lambda_1 e^{\delta t}$ and $\lambda_2 e^{\delta t}$ does not depend on time t in an optimal equilibrium. Therefore, they are bounded when t tend to ∞.

6 Scenario of delay model

In section 6, we shall discuss the dynamical behaviour of the delayed model system (2.5). Delay can produce very interesting population phenomena in the population model. The differential equations in the ecological model system with delay give rise to delay differential equations. Mathematical analysis shows the stable as well as unstable periodic oscillation of the populations. To investigate the effect of delay in the present population model we consider the most relevant equilibrium point (x_2, y_2) of the system (2.5). We take the transformation $X = x - x_2$, $Y = y - y_2$. Substituting these into the system of equations (2.5) and then using conditions of equilibrium we get,

$$\begin{aligned} \frac{dX}{dt} &= (\alpha - 2\beta x_2 - \mu\gamma\frac{y_2}{(\mu + x_2)^2} - E_1)X - \frac{\gamma x_2}{\mu + x_2}Y \\ \frac{dY}{dt} &= \frac{\gamma_1 \mu y_2}{(\mu + x_2)^2}X(t-\tau) + (\frac{\gamma x_2}{\mu + x_2} - d - 2\beta_1 y - E_2)Y \end{aligned} \quad (6.1)$$

6.1 Criteria for preservation of delay induced stability

The approximated linear system of the nonlinear system (2.5) gives the exact behaviour of the nonlinear system [18]. So, we take less effort to study with the linearised system (6.1). The characteristic equation of (6.1) (linearised system) is

$$F(\lambda, \tau) = 0, \quad (6.2)$$

where $F(\lambda,\tau)=\lambda^2-P\lambda+Q+Re^{-\lambda\tau}$. Here $P=(\alpha-2\beta x_2-\mu\gamma\frac{y_2}{(\mu+x_2)^2}-E_1)+(\frac{\gamma x_2}{\mu+x_2}-d-2\beta_1 y-E_2)$

$Q=(\alpha-2\beta x_2-\mu\gamma\frac{y_2}{(\mu+x_2)^2}-E_1)(\frac{\gamma x_2}{\mu+x_2}-d-2\beta_1 y-E_2)$, $R=\frac{\gamma\gamma_1\mu x_2 y_2}{(\mu+x_2)^3}$.

The steady state is asymptotically stable when the equation (6.2) will have two roots with negative real-part. Now we shall use the theorem given by Gopalsamy [21] to get the conditions for the non existence of delay-induced instability.

Theorem: An equilibrium point $P_2(x_2,y_2)$ is locally asymptotically stable in presence of time delay τ iff

(i) All roots of $F(\lambda,0)=0$ will have negative real-part,

(ii) For any ω and $\tau>0$, $F(i\omega,\tau)\neq 0$, where ω ir real and $i=\sqrt{-1}$.

Proof: (i)When $\tau=0$

We have already found the criteria for which the roots of the equation are negative,that is

$\frac{\gamma x_2}{(\mu+x_2)^2}-\frac{\beta x_2}{y_2}<\beta_1<\frac{\gamma\gamma_1\mu}{(\mu+x_2)(\gamma y_2-\beta(\mu+x_2)^2}$ and $\frac{\beta}{\gamma}<\frac{y_2}{(\mu+x_2)^2}$.

(ii) Let $\tau\neq 0$. Putting $\lambda=i\omega$ in (6.2) (ω is real) we get,

$F(i\omega,\tau)=(-\omega^2+Q+Rcos\omega\tau)-i(P\omega+Rsin\omega\tau)$.

If $\omega=0$, then $F(0,\tau)=Q\neq 0$.

If $\omega\neq 0$, then if possible let $i\omega$ satisfies the characteristic equation.

Then we get, $F(i\omega,\tau)=(-\omega^2+Q+Rcos\omega\tau)-i(P\omega+Rsin\omega\tau)=0$.

Comparing we get,

$-\omega^2+Q+Rcos\omega\tau=0$ and $P\omega+Rsin\omega\tau=0$

$$\omega^2-Q = Rcos\omega\tau \qquad (6.3)$$

$$-P\omega = Rsin\omega\tau \qquad (6.4)$$

Squaring and adding, we get

$$\omega^4+(P^2-2Q)\omega^2+Q^2-R^2=0. \qquad (6.5)$$

By Descartes's rule of sign, if $P^2-2Q>0$ and $Q^2-R^2>0$, then the above equation has no positive values of ω^2 i.e. $\omega^2<0$, which implies ω is imaginary which is a contradiction since ω is real. Therefore, $i\omega$ is not a root of $F(\lambda, \tau) = 0$. Therefore $F(i\omega, \tau) \neq 0$ when $\tau \neq 0$ and for every real ω. Also $P^2 - 2Q > 0$ and $Q^2 - R^2 > 0$ implies $P^4 > 4R^2$. Thus the nonexistence of delay-induced instability of the delay model (6.1) is given by $P^4 > 4R^2$. With this analysis, the following theorem for delay-induced instability of the system can be stated:

Theorem: In presence of time delay the system is locally asymptotically stable if $\frac{\gamma x_2}{(\mu+x_2)^2} - \frac{\beta x_2}{y_2} < \beta_1 < \frac{\gamma\gamma_1\mu}{(\mu+x_2)(\gamma y_2-\beta(\mu+x_2)^2)}$, $\frac{\beta}{\gamma} < \frac{y_2}{(\mu+x_2)^2}$ and $P^4 > 4R^2$.

6.2 Oscillatory phenomena

We have shown that the delay model is stable near $P_2(x_2, y_2)$ if the above theorem is satisfied. Now we shall analyze whether the time delay can make a stable system into an unstable one. If the characteristic equation posses two purely imaginary roots, then a stable equilibrium point will become an unstable one. Let us consider the time delay τ as a bifurcation parameter and λ be a function of τ. If $\lambda = i\omega(\tau)$ is purely imaginary root of (6.2) for some value of $\tau = \tau^*$ and let $\omega(\tau^*) = \omega^* \neq 0$. ω^* can be obtained from the equation (6.3). If $P^2 - 2Q > 0$ and $Q^2 - R^2 < 0$ then there exists exactly one positive root ω^{*2} of the equation (6.5).

τ^* is given by $\tau^* = \frac{1}{\omega^*}tan^{-1}\frac{Q\omega^*}{Q-\omega^{*2}} + \frac{n\pi}{\omega^*}$ where n = 0,1,2,...

For n=0, $\tau_0^* = \frac{1}{\omega^*}tan^{-1}\frac{Q\omega^*}{Q-\omega^{*2}}$ which is the smallest time delay.

Therefore, there is $(i\omega^*, \tau^*)$ which can change the stability of the delay model around $P_2(x_2, y_2)$. Now we want to investigate whether Hopf-bifurcation occurs at this point. The transversality condition

$$\frac{d}{d\tau}Re(\lambda)|_(i\omega^*, \tau^*) \neq 0.$$

will be verified.

To verify the above criterion we need only the sign of $\frac{d}{d\tau}Re(\lambda)|_(i\omega^*, \tau^*)$. Again, the sign of $\frac{d}{d\tau}Re(\lambda)|_(i\omega^*, \tau^*)$ is same as sign of $Re[\frac{d\lambda}{d\tau}]^{-1}|_(i\omega^*, \tau^*)$.

Differentiating the characteristic equation and rearranging we get, $Re[\frac{d\lambda}{d\tau}]^{-1}|_(i\omega^*, \tau^*) = \frac{2((\omega^*)^2-Q)+P^2}{R^2}$.

Let $s = \omega^2$, then the left hand side of (6.5) becomes $G = s^2 + (P^2 - 2Q)s + (Q^2 - R^2)$. Differentiating, we get

$\frac{dG}{ds}|_{(s=\omega^{*2})} = 2(\omega^{*2} - Q) + P^2$.

Therefore,

$$Re[\frac{d\lambda}{d\tau}]^{-1}|_{(}i\omega^*, \tau^*) = \frac{1}{R^2}\frac{dG}{ds}|_{(s=\omega^{*2})}.$$

As the characteristic equation can not have a multiple imaginary roots, therefore

$$Re[\frac{d\lambda}{d\tau}]^{-1}|_{(}i\omega^*, \tau^*) = \frac{1}{R^2}\frac{dG}{ds}|_{(s=\omega^{*2})} \neq 0.$$

Therefore, the transversality condition of Hopf-bifurcation is satisfied. Therefore, a Hopf-bifurcation point is $(i\omega^*, \tau^*)$.

The above analysis shows that the system undergoes Hopf-bifurcation under certain conditions. The stable trajectories of the delay differential equations system given by (2.5) reduce to periodic oscillation for attaining some value of the gestational delay parameter $\tau = \tau^*$. When $\tau < \tau^*$, the system trajectories are bounded in a region in the plane of the predator and prey, whereas the trajectories become periodic when $\tau >= \tau^*$. Thus the gestational delay can bring the system into an unstable one.

7 Discussion of the results and conclusion

We have taken a predator-prey model under harvesting and the functional response is taken as Holling type II functional response. There are analyses of the predator-prey system where Holling type II [6] and even Holling type III functional response [22] have been used. To discuss the qualitative behaviour of the model, they have found some conditions or restrictions on the parameters of the system. In the present model, we have introduced self-crowding of the predator species, which plays a vital role in regulating the ecosystem. The effect of self-crowding was studied in Kobras et al. [23]. We have shown how the crowding of the predator population in the model system affects the stability.

In the present model, harvesting is a very important phenomenon in the ecological system. The activities are being regulated by humanbeing and the exploitation of the species is very common. In this situation, the system population would become extinct from the ecosystem in some time.

In this model, we get some conditions under which the steady states are stable. These conditions mainly depend on the growth rate, death rate, harvesting effort and competition among the predator population. We have shown that, to maintain the stability of the coexistence of both species, the competition among the predators must follow some parametric conditions as described above. However, the model with some prescribed parametric restrictions undergoes periodic oscillation.

We have solved the system taking numerical values of the parameters as follows: $\alpha = 38.5$; $\beta = 0.28$; $\gamma = 5.0$; $\mu = 2.8$; $E1 = 30.5$; $\gamma_1 = 4.3$; $d = 0.75$; $\beta_1 = 0.114$; $E2 = 1.05$. The solution of the system is stable as shown in the figure 1. Then the system evolves a periodic oscillation around $(16.8560, 14.0882)$ for the change of the parameter $\beta_1 = 0.117$ from $\beta_1 = 0.114$. The numerical solution is shown in figure 2. In this case the value of the Liapunov exponent is $\sigma = -1.7419$ which is negative. Thus the periodic oscillation is stable.

The existence of the bionomic equilibrium of the non-delayed system is examined. We have used Pontryagin's maximum principle [24] to study optimal harvesting policy.

The present value J over some time interval $[0, \infty]$ is optimized subject to the control constraints and the state equations. We have shown that shadow prices is constant and economic revenue is at its maximum when the discount rate is zero. The economic rate is wholly dissipated for infinite discount rate.

We have considered the model system in which discrete type time delay is present. The analysis of delay model system is difficult than the non-delayed system. We have found criteria for which equilibrium points exists and they are stable also find some conditions to exist bifurcations of the delayed model. The time delay has the ability to derive a stable equilibrium point to an unstable one. Our analysis shows the affirmation

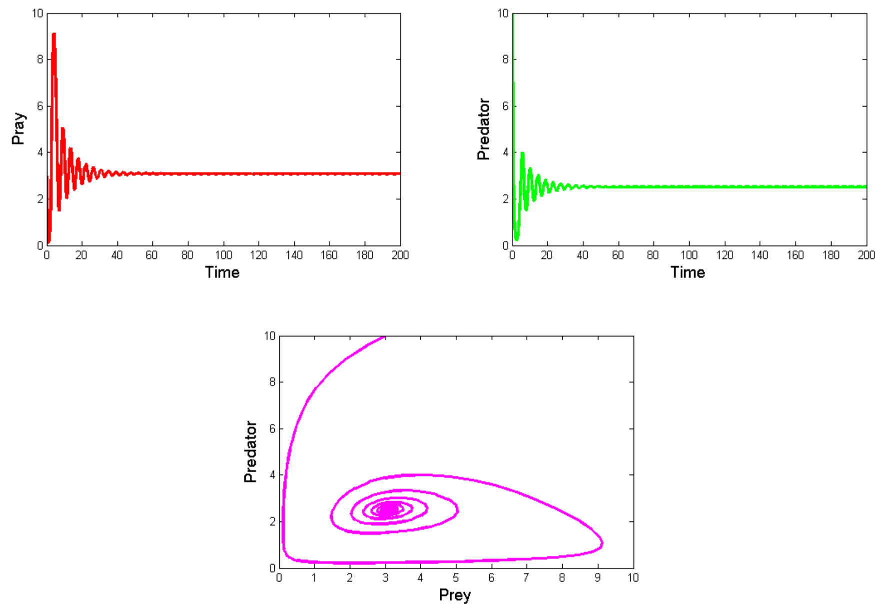

Figure 1: Asymptotically convergence of prey and predator population and Phase portrait of the non-delayed system.

of the delay system's instability. The system remains stable when the delay parameter assume some values. We have determined the smallest value of that parameter for which the system remains stable.

The model system has been numerically solved using MATLAB routine. Taking the parameter values as follows: $\alpha = 38.5, \beta = 0.28, \gamma = 5.0, E_1 = 35.5, \gamma_1 = 4.3, d = 0.75, \beta_1 = 0.18, E_2 = 1.05, \mu = 2.8,\ \tau = 0$, we get a stable solution of the two species which strike the equilibrium level $(3.0793, 2.5199)$. The solution is depicted in the figure 3.

The model system without delay exhibits periodic oscillation for continuous change of the parameter β_1.

Now, keeping all parameter values the same, we have changed the value of τ (gestational delay parameter) and we have shown that for $\tau = 0.08$, the system remains asymptotically stable. It is depicted in the figure 2. Once the value of the delay parameter is increased to $\tau = 0.088$, the behaviour of the system is drastically changed, and

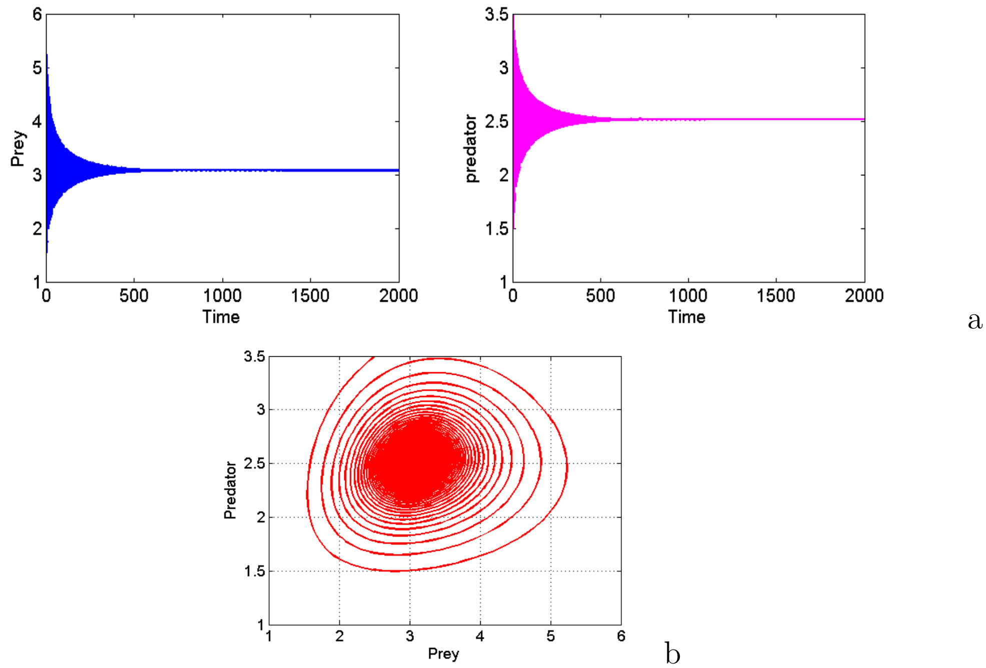

Figure 2: (a) Asymptotic stability of the delayed system for delay $\tau = 0.08$, (b) Phase portrait of the delayed system for delay $\tau = 0.08$.

the orbits become periodic, as shown in figure 4.

Thus the numerical solutions agree with the analytical solutions of the system as described in the previous sections. The crowding of the predator population transmits responses to the system which changes the dynamical behaviour.

The system we have proposed is some modification of other form which is studied earlier. Use of such modified form has advantages because exact interactions among the ecological systems are unknown. The examination of the modified system is expected to produce results that may be applicable to certain model classes. The population dynamics become chaotic when interacting with the ecological population. On the other hand the chaotic dynamics lead to the destabilization of the model.

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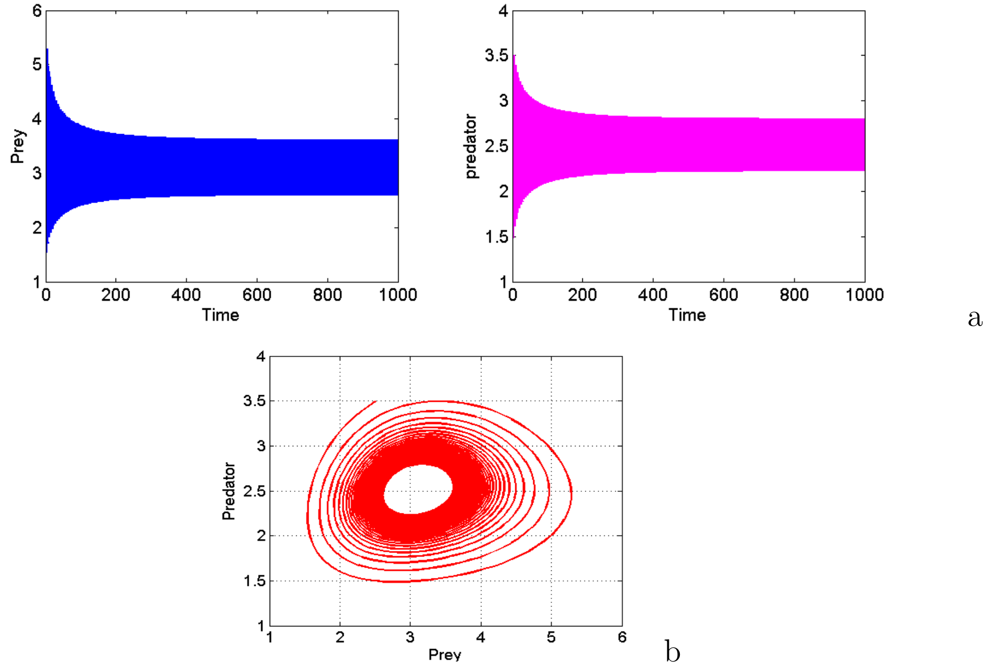

Figure 3: (a) Periodic time series for the delayed system for delay $\tau = 0.088$, (b) Periodic phase portrait of the delayed system for delay $\tau = 0.088$.

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